

# Logics of Finite Hankel Rank

(For Yuri Gurevich at his 75th Birthday)

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## Abstract

We discuss the Feferman-Vaught Theorem in the setting of abstract model theory for finite structures. We look at sum-like and product-like binary operations on finite structures and their Hankel matrices. We show the connection between Hankel matrices and the Feferman-Vaught Theorem. The largest logic known to satisfy a Feferman-Vaught Theorem for product-like operations is CFOL, first order logic with modular counting quantifiers. For sum-like operations it is CMSOL, the corresponding monadic second order logic. We discuss whether there are maximal logics satisfying Feferman-Vaught Theorems for finite structures.

## 1 Introduction

### 1.1 Yuri's Quest for Logics for Computer Science

The second author (JAM) first met Yuri Gurevich in spring 1976, while being a Lady Davis fellow at the Hebrew University, on leave from the Free University, Berlin. Yuri had just recently emigrated to Israel. Yuri was puzzled by the supposed leftist views of JAM, perceiving them as antagonizing. This led to heated political discussions. In the following time, JAM spent more visiting periods in Israel, culminating in the Logic Year of 1980/81 at the Einstein Institute of the Hebrew University, after which he finally joined the Computer Science Department at the Technion in Haifa.

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At this time both Yuri and JAM worked on chapters to be published in [1], Yuri on Monadic Second Order Logic, and JAM on abstract model theory. Abstract model theory deals with meta-mathematical characterizations of logic. Pioneered by P. Lindström, G. Kreisel and J. Barwise, in [1, 2], [26], [30, 31], First Order Logic and admissible fragments of infinitary logic were characterized. Inspired by H. Scholz's problem, [10], R. Fagin initiated similar characterizations when models are restricted to finite models, connecting finite model theory to complexity theory.

At about the same time Yuri and JAM both underwent a transition in research orientation, slowly refocusing on questions in theoretical Computer Science. Two papers document their evolving views at the time, [23], [37]. Yuri was vividly interested in [37] and frequent discussions between Yuri and JAM between 1980 and 1982 shaped both papers. In [37] the use for theoretical computer science of classical model theoretic methods, in particular, the role of the classical preservation theorems (see below), was explored, see also [34], [36]. Yuri grasped early on that these preservation theorems do not hold when one restricts First Order Logic to finite models.

Under the influence of JAM's work in abstract model theory, the foundations of database theory and logic programming, [5, 6], [34], [36], [38], [43], and the work of N. Immerman and M. Vardi, [24], [47], Yuri stressed the difference between classical model theory and finite model theory. In [23], he formulated what he calls the Fundamental Problem of finite model theory. This problem is, even after 30 years, still open ([23]): Is there a logic  $\mathcal{L}$  such that any class  $\Phi$  of finite structures is definable in  $\mathcal{L}$  iff  $\Phi$  is recognizable in polynomial time. For *ordered* finite structures there are several such logics, [21], [24], [39], [41, 42], [47]. We give a precise statement of the Fundamental Problem in [section 2](#), [Problem 1](#).

## 1.2 Preservation Theorems

Let  $\mathcal{F}_1, \mathcal{F}_2$  be two syntactically defined fragments of a logic  $\mathcal{L}$ , and let  $R$  be a binary relation between structures. *Preservation theorems* are of the form:

Let  $\phi \in \mathcal{F}_1$ . The following are equivalent:

- (i) For all structures  $\mathfrak{A}, \mathfrak{B}$  with  $R(\mathfrak{A}, \mathfrak{B})$ , we have that if  $\mathfrak{A}$  satisfies  $\phi_1 \in \mathcal{F}_1$ , then also  $\mathfrak{B}$  satisfies  $\phi_1$ .
- (ii) There is  $\phi_2 \in \mathcal{F}_2$  which is logically equivalent to  $\phi_1$ .

A typical example is Tarski's Theorem for first order logic, with  $\mathcal{F}_1$  all of first order logic,  $\mathcal{F}_2$  its universal formulas, and  $R(\mathfrak{A}, \mathfrak{B})$  holds if  $\mathfrak{B}$  is a substructure of  $\mathfrak{A}$ . Many other preservation theorems can be found in [7]. In response to [37], [43], Yuri pointed out in [23] that most of the preservation theorems for first order logic fail when one restricts models to be finite.

### 1.3 Reduction Theorems

Let  $(\mathcal{F}_2)^*$  denote the finite sequences of formulas in  $\mathcal{F}_2$ , and let  $\square$  be a binary operation on finite structures. *Reduction theorems* are of the form:

There is a function  $p : \mathcal{F}_1 \rightarrow (\mathcal{F}_2)^*$  with  $p(\phi) = (\psi_1, \dots, \psi_{2 \cdot k(\phi)})$  and a Boolean function  $B_\phi$  such that for all structures  $\mathfrak{A} = \mathfrak{B}_1 \square \mathfrak{B}_2$  and all  $\phi \in \mathcal{F}_1$ , the structure  $\mathfrak{A}$  satisfies  $\phi \in \mathcal{F}_1$  iff

$$B(\psi_1^{B_1}, \dots, \psi_{k(\phi)}^{B_1}, \psi_1^{B_2}, \dots, \psi_{k(\phi)}^{B_2}) = 1 \quad (1)$$

where for  $1 \leq j \leq k$  we have  $\psi_j^{B_1} = 1$  iff  $\mathfrak{B}_1 \models \psi_j$  and  $\psi_j^{B_2} = 1$  iff  $\mathfrak{B}_2 \models \psi_j$ . There are also versions for  $(n)$ -ary operations  $\square$ .

The most famous examples of such reduction theorems are the Feferman-Vaught-type theorems, [12, 13, 14, 15], [22], [40]. A simple case is Monadic Second Order Logic (MSOL), where  $\mathcal{F}_1 = \mathcal{F}_2 = \text{MSOL}$  and  $\mathfrak{A}$  is the disjoint union  $\sqcup$  of  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$ . Additionally it is required that the quantifier ranks of the formulas in  $p(\phi)$  do not exceed the quantifier rank of  $\phi$ . In [38, Chapter 4] such reduction theorems are discussed in the context of abstract model theory. However, in [38, Chapter 4] the quantifier rank has no role.

In contrast to preservation theorems, reduction theorems still hold when restricted to finite structures.

### 1.4 Purpose of this Paper

In [40] JAM discussed Feferman-Vaught-type theorems in finite model theory and their algorithmic uses. In Section 7 of that paper, it was asked whether one can characterize logics over finite structures which satisfy the Feferman-Vaught Theorem for the disjoint union  $\sqcup$ . The purpose of this paper is to outline new directions to attack this problem. The novelty in our approach is in relating the Feferman-Vaught Theorem to Hankel matrices of sum-like and connection-like operations on finite structures. Hankel matrices for connection-like operations, aka connection matrices, have many algorithmic applications, cf. [28], [33].

In [section 2](#) we set up the necessary background on Lindström logics, quantifier rank, translation schemes, and sum-like operations. A Hankel matrix  $H(\Phi, \square)$  involves a binary operation  $\square$  on finite  $\sigma$ -structures which results in a  $\tau$ -structure, and a class of  $\tau$ -structures  $\Phi$  closed under isomorphisms (aka a  $\tau$ -property). In [section 3](#) we give the necessary definitions of Hankel matrices and their rank. We then study  $\tau$ -properties  $\Phi$  where  $H(\Phi, \square)$  has finite rank. We show that there are uncountably many such properties and state that the class of all properties that have finite rank for every sum-like operation  $\square$  forms a Lindström logic, Theorems [5](#) and [8](#). In [section 4](#) we define various forms of Feferman-Vaught-type properties of Lindström logics equipped with a quantifier rank, and discuss their connection to Hankel matrices. [Theorem 16](#) describes their exact relationship. A logic has finite S-rank, if all its definable  $\tau$ -properties have Hankel matrices of finite rank for every sum-like operation. In [section 5](#) we sketch how to construct a logic satisfying the Feferman-Vaught Theorem for

sum-like operations from a logic which has finite S-rank. Finally, in [section 6](#), we discuss our conclusions and state open problems. A full version of this paper is in preparation, [\[27\]](#).

## 2 Background

### 2.1 Logics with Quantifier Rank

We assume the reader is familiar with the basic definitions of generalized logics, see [\[1\],\[11\]](#). We denote by  $\tau$  finite relational vocabularies, possibly with constant symbols for named elements.  $\tau$ -structures are always *finite* unless otherwise stated. A finite structure of size  $n$  is always assumed to have as its universe the set  $[n] = \{1, \dots, n\}$ . A class of finite  $\tau$ -structures  $\Phi$  closed under  $\tau$ -isomorphisms is called a  $\tau$ -*property*.

A *Lindström Logic*  $\mathcal{L}$  is a triple

$$\langle \mathcal{L}(\tau), \text{Str}(\tau), \models_{\mathcal{L}}, \rangle$$

where  $\mathcal{L}(\tau)$  is the set of  $\tau$ -sentences of  $\mathcal{L}$ ,  $\text{Str}(\tau)$  are the *finite*  $\tau$ -structures,  $\models_{\mathcal{L}}$  is the satisfaction relation. The satisfaction relation is a ternary relation between  $\tau$ -structures, assignments and formulas. An *assignment* for variables in a  $\tau$ -structure  $\mathfrak{A}$  is a function which assigns to each variable an element of the universe of  $\mathfrak{A}$ . We always assume that the logic contains all the atomic formulas with free variables, and is closed under Boolean operations and first order quantifications. A logic  $\mathcal{L}_0$  is a *sublogic* of a logic  $\mathcal{L}$  iff  $\mathcal{L}_0(\tau) \subseteq \mathcal{L}(\tau)$  for all  $\tau$  and the satisfaction relation of  $\mathcal{L}_0$  is the satisfaction relation induced by  $\mathcal{L}$ .

A *Gurevich logic*  $\mathcal{L}$  is a Lindström logic where additionally the sets  $\mathcal{L}(\tau)$  are uniformly computable.

A *Lindström logic*  $\mathcal{L}$  with a *quantifier rank* is a quadruple

$$\langle \mathcal{L}(\tau), \text{Str}(\tau), \models_{\mathcal{L}}, \rho \rangle$$

where additionally  $\rho$  is a *quantifier rank function*. A quantifier rank (q-rank)  $\rho$  is a function  $\rho : \mathcal{L}(\tau) \rightarrow \mathbb{N}$  such that

- (i) For atomic formulas  $\phi$  the q-rank  $\rho(\phi) = 0$ .
- (ii) Boolean operations and translations induced by translation schemes (see [subsection 2.2](#)) with formulas of q-rank 0 preserve maximal q-rank.

A quantifier rank  $\rho$  is *nice* if additionally it satisfies the following:

- (iii) For finite  $\tau$ , there are, up to logical equivalence, only finitely many  $\mathcal{L}(\tau)$ -formulas of fixed q-rank with a fixed set of free variables.

In the presence of (iii) we define *Hintikka formulas* as maximally consistent  $\mathcal{L}(\tau)$ -formulas of fixed q-rank. A *nice logic*  $\mathcal{L}$  is Lindström logic with a nice

quantifier rank  $\rho$ . We note that in a nice logic, the only formulas  $\phi$  of q-rank  $\rho(\phi) = 0$  are Boolean combinations of atomic formulas.

We denote by FOL, MSOL, SOL, first order, monadic second order, and full second order logic, respectively. All these logics are nice Gurevich logics with their natural quantifier rank, and they are sublogics of SOL.

We denote by CFOL, CMSOL, first order and monadic second order logic augmented by the modular counting quantifiers  $D_{k,m}x.\phi(x)$  which say that there are modulo  $m$ , exactly  $k$  many elements satisfying  $\phi$ . In the presence of the quantifier  $D_{k,m}$  there are two definitions of the quantifier rank:  $\rho_1(D_{k,m}x.\phi(x)) = 1 + \rho_1(\phi)$  and  $\rho_2(D_{k,m}x.\phi(x)) = m + \rho_2(\phi)$ . Given any finite set of variables, for  $\rho_1$  we have, up to logical equivalence, infinitely many formulas  $\phi$  with  $\rho_1(\phi) = 1$ , whereas for  $\rho_2$  there are only finitely many such formulas. CFOL and CMSOL with the quantifier rank  $\rho_2$  are nice Gurevich logics. In the sequel we always use  $\rho_2$  as the quantifier rank for CFOL and CMSOL.

FPL, fixed point logic, is also a Gurevich logic and a sublogic of SOL. However, *order invariant* FPL is a sublogic of SOL which is not a Lindström logic. The definable  $\tau$ -properties in order invariant FPL are exactly the  $\tau$ -properties recognizable in polynomial time. For FPL and order invariant FPL see [21],[24],[39],[41, 42],[47].

**Problem 1** (Y. Gurevich, [23]). *Is there a Gurevich logic  $\mathcal{L}$  such that the  $\mathcal{L}$ -definable  $\tau$ -properties are exactly the  $\tau$ -properties recognizable in polynomial time.*

## 2.2 Sum-Like and Product-Like Operations on $\tau$ -structures

The following definitions are taken from [40]. Let  $\tau, \sigma$  be two relational vocabularies with  $\tau = \langle R_1, \dots, R_m \rangle$ , and denote by  $r(i)$  the arity of  $R_i$ . A  $(\sigma - \tau)$  translation scheme  $T$  is a sequence of  $\mathcal{L}(\sigma)$ -formulas  $(\phi; \phi_1, \dots, \phi_m)$  where  $\phi$  has  $k$  free variables, and each  $\phi_i$  has  $k \cdot r(i)$  free variables. In this paper we do not allow redefining equality, nor do we allow name changing of constants.

We associate with  $T$  two mappings  $T^* : \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$  and  $T^\sharp : \mathcal{L}(\tau) \rightarrow \mathcal{L}(\sigma)$ , the *transduction and translation induced by  $T$* . The transduction of a  $\sigma$ -structure  $\mathfrak{A}$  is the  $\tau$ -structure  $T^*(\mathfrak{A})$  where the vocabulary is interpreted by the formulas given in the translation scheme. The translation of a  $\tau$ -formula is obtained by substituting atomic  $\tau$ -formulas with their definition through  $\sigma$ -formulas given by the translation scheme. A translation scheme (induced transduction, induced translation) is *scalar* if  $k = 1$ , otherwise it is *k-vectorized*. It is *quantifier-free* if so are the formulas  $\phi; \phi_1, \dots, \phi_m$ .

If  $\tau$  has no constant symbols, the disjoint union  $\mathfrak{A} \sqcup \mathfrak{B}$  of two  $\tau$ -structures  $\mathfrak{A}, \mathfrak{B}$  is the  $\tau$ -structure obtained by taking the disjoint union of the universes and of the corresponding relation interpretations in  $\mathfrak{A}$  and  $\mathfrak{B}$ . On the other hand, if  $\tau$  has finitely many constant symbols  $a_1, \dots, a_k$  the disjoint union of two  $\tau$ -structures is a  $\tau'$ -structure with twice as many constant symbols,  $\tau' = \tau \cup \{a'_1, \dots, a'_k\}$ . *Connection operations* are similar to disjoint unions

with constants, where equally named elements are identified. We call the disjoint union followed by the pairwise identification of  $k$  constant pairs the  $k$ -sum, cf. [33].

A binary operation  $\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$  is *sum-like* (*product-like*) if it is obtained from the disjoint union of  $\sigma$ -structures by applying a quantifier-free scalar (vectorized)  $(\sigma - \tau)$ -transduction. A binary operation  $\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$  is *connection-like* if it is obtained from a connection operation on  $\sigma$ -structures by applying a quantifier-free scalar  $(\sigma - \tau)$ -transduction. If  $\sigma = \tau$ , we say  $\square$  is an operation on  $\tau$ -structures.

Connection-like operations are not sum-like according to the definitions in this paper<sup>1</sup>. Although connection operations are frequently used in the literature, cf. [33],[40], we do not deal with them in this paper. Most of our results here can be carried over to connection-like operations, but the formalism required to deal with the identification of constants is tedious and needs more place than available here.

**Proposition 1.** *Let  $\tau$  be a fixed finite relational vocabulary.*

- (i) *There are only finitely many sum-like binary operations on  $\tau$ -structures.*
- (ii) *There is a function  $\alpha : \mathbb{N} \rightarrow \mathbb{N}$  such that for each  $k \in \mathbb{N}$  there are only  $\alpha(k)$  many  $k$ -vectorized product-like binary operations on  $\tau$ -structures.*

## 2.3 Abstract Lindström Logics

In [31] a syntax-free definition of a logic is given. An *abstract Lindström logic*  $\mathcal{L}$  consists of a family  $\text{Mod}(\tau)$  of  $\tau$ -properties closed under certain operations between properties of possibly different vocabularies. One thinks of  $\text{Mod}(\tau)$  as the family of  $\mathcal{L}$ -definable  $\tau$ -properties. We do not need all the details here, the reader may consult [1],[30, 31]. The main point we need is that every abstract Lindström logic  $\mathcal{L}$  can be given a canonical syntax  $\mathcal{L}(\tau)$  using generalized quantifiers.

## 3 Hankel matrices of $\tau$ -properties

### 3.1 Hankel Matrices

In linear algebra, a *Hankel matrix*, named after Hermann Hankel, is a real or complex square matrix with constant skew-diagonals. In automata theory, a *Hankel matrix*  $H(f, \circ)$  is an infinite matrix where the rows and columns are labeled with words  $w$  over a fixed alphabet  $\Sigma$ , and the entry  $H(f, \circ)_{u,v}$  is given by  $f(u \circ v)$ . Here  $f : \Sigma^* \rightarrow \mathbb{R}$  is a real-valued word function and  $\circ$  denotes concatenation. A classical result of G.W. Carlyle and A. Paz [4] in automata theory characterizes real-valued word functions  $f$  recognizable by weighted (aka multiplicity) automata in algebraic terms.

<sup>1</sup>They are nevertheless called sum-like in [40].

Hankel matrices for graph parameters (aka connection matrices) were introduced by L. Lovász [32] and used in [18],[33] to study real-valued partition functions of graphs. In [18],[33] the role of concatenation is played by  $k$ -connections of  $k$ -graphs, i.e., graphs with  $k$  distinguished vertices  $v_1, \dots, v_k$ .

In this paper we study  $(0, 1)$ -matrices which are Hankel matrices of properties of general relational  $\tau$ -structures and the role of  $k$ -connections is played by more general binary operations, the sum-like and product-like operations introduced in [44] and further studied in [40].

**Definition 2.** Let  $\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$  be a binary operation on finite  $\sigma$ -structures returning a  $\tau$ -structure, and let  $\Phi$  be a  $\tau$ -property.

- (i) The Boolean Hankel matrix  $H(\Phi, \square)$  is the infinite  $(0, 1)$ -matrix where the rows and columns are labeled by all the finite  $\sigma$ -structures, and  $H(\Phi, \square)_{\mathfrak{A}, \mathfrak{B}} = 1$  iff  $\mathfrak{A} \square \mathfrak{B} \in \Phi$ .
- (ii) The rank of  $H(\Phi, \square)$  over  $\mathbb{Z}_2$  is denoted by  $r(\Phi, \square)$ , and is referred to as the Boolean rank.
- (iii) We say that  $\Phi$  has finite  $\square$ -rank iff  $r(\Phi, \square)$  is finite.
- (iv) Two  $\sigma$ -structures are  $(\Phi, \square)$ -equivalent,  $\mathfrak{A} \equiv_{\Phi, \square} \mathfrak{B}$ , if for all finite  $\sigma$ -structures  $\mathfrak{C}$  we have

$$\mathfrak{A} \square \mathfrak{C} \in \Phi \text{ iff } \mathfrak{B} \square \mathfrak{C} \in \Phi \quad (2)$$

- (v) For a  $\sigma$ -structure  $\mathfrak{A}$ , we denote by  $[\mathfrak{A}]_{\Phi, \square}$  the  $(\Phi, \square)$ -equivalence class of  $\mathfrak{A}$ .
- (vi) We say that  $\Phi$  has finite  $\square$ -index<sup>2</sup> iff there are only finitely many  $(\Phi, \square)$ -equivalence classes.

**Proposition 3.** Let  $\Phi$  be a  $\tau$ -property.  
 $\Phi$  has finite  $\square$ -rank iff  $\Phi$  has finite  $\square$ -index.

*Sketch of proof.* We first note that two  $\sigma$ -structures  $\mathfrak{A}, \mathfrak{B}$  are in the same equivalence class of  $\equiv_{\Phi, \square}$  iff they have identical rows in  $H(\Phi, \square)$ . As the rank is over  $\mathbb{Z}_2$ , finite rank implies there are only finitely many different rows in  $H(\Phi, \square)$ . The converse is obvious. Q.E.D.

### 3.2 $\tau$ -Properties of Finite $\square$ -rank

We next show that there are uncountably many  $\tau$ -properties of finite  $\sqcup$ -rank. We also study the relationship between the  $\square_1$ -rank and  $\square_2$ -rank of  $\tau$ -properties for different operations  $\square_1$  and  $\square_2$ .

We first need a lemma.

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<sup>2</sup> K. Compton and I. Gessel, [8],[19], already considered  $\tau$ -properties of finite  $\sqcup$ -index for the disjoint union of  $\tau$ -structures. In [17] this is called *Gessel index*. C. Blatter and E. Specker, in [3],[46], consider a substitution operation on pointed  $\tau$ -structures,  $\text{Subst}(\mathfrak{A}, a, \mathfrak{B})$ , where the structure  $\mathfrak{B}$  is inserted into  $\mathfrak{A}$  at a point  $a$ .  $\text{Subst}(\mathfrak{A}, a, \mathfrak{B})$  is sum-like, and the *Subst-index* is called in [17] *Specker index*.

**Lemma 4.** Let  $A \subseteq \mathbb{N}$  and let  $M_A$  be the infinite  $(0,1)$ -matrix whose columns and rows are labeled by the natural numbers  $\mathbb{N}$ , and  $(M_A)_{i,j} = 1$  iff  $i + j \in A$ . Then  $M_A$  has finite rank over  $\mathbb{Z}_2$  iff  $A$  is ultimately periodic.

**Theorem 5.** Let  $\tau_{\text{graphs}}$  be the vocabulary with one binary edge-relation, and  $\tau_1$  be  $\tau_{\text{graphs}}$  augmented by one vertex label. Let  $C_A, \overline{C_A}$  and  $P_A$  be the graph properties defined by  $C_A = \{K_n : n \in A\}$ ,  $\overline{C_A} = \{E_n : n \in A\}$ , and  $P_A = \{P_n : n \in A\}$ , where  $E_n$  is the complement graph of the clique  $K_n$  of size  $n$ , and  $P_n$  is a path graph of size  $n$ .

(i)  $H(C_A, \sqcup)$  has finite rank for all  $A \subseteq \mathbb{N}$ .

(ii) For two graphs  $G_1, G_2$ , let  $G_1 \sqcup^c G_2$  be the sum-like operation defined as the loopless complement graph of  $G_1 \sqcup G_2$ .

$H(C_A, \sqcup^c)$  has infinite rank for all  $A \subseteq \mathbb{N}$  which are not ultimately periodic.

Equivalently, for the  $\tau_{\text{graphs}}$ -property  $\overline{C_A}$ , the Hankel matrix  $H(\overline{C_A}, \sqcup)$  has infinite rank for all  $A \subseteq \mathbb{N}$  which are not ultimately periodic.

(iii)  $H(P_A, \square)$  has finite rank for all sum-like operations  $\square$  on  $\tau_{\text{graphs}}$ -structures and all  $A \subseteq \mathbb{N}$ .

(iv) For two graphs  $G_1, G_2$  with one vertex label, i.e.  $\tau_1$ -structures, let  $G_1 \sqcup^1 G_2$  be the sum-like operation defined as the graph resulting from  $G_1 \sqcup G_2$  by adding an edge between the two labeled vertices and then removing the labels.  $H(P_A, \sqcup^1)$  has infinite rank for all  $A \subseteq \mathbb{N}$  which are not ultimately periodic.

(v)  $H(C_A, \sqcup_k)$  has finite rank for all  $A \subseteq \mathbb{N}$ .

**Theorem 5** needs an interpretation: (i) says that there is a specific sum-like operation  $\square$  such that there uncountably many classes of  $\tau$ -structures with finite  $\square$ -rank<sup>3</sup>. (ii) says that if a class has finite  $\square$ -rank for one sum-like operation, it does not have to hold for all sum-like operations<sup>4</sup>. (iii) produces uncountably many classes of  $\tau$ -structures which have finite  $\square$ -rank for all sum-like operations on  $\tau$ -structures. (iv) finally shows that such classes can still have infinite  $\square$ -rank for sum-like operations which take as inputs  $\sigma$ -structures (labeled paths) and output a  $\tau$ -structure (unlabeled paths). This leads us to the following definition:

**Definition 6.** Let  $\tau$  be a vocabulary and  $\Phi$  be a  $\tau$ -property.

(i)  $\Phi$  has finite  $S$ -rank ( $P$ -rank,  $C$ -rank) if for every sum-like (product-like, connection-like) operation  $\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$  the Boolean rank of  $H(\Phi, \square)$  is finite.

<sup>3</sup> A similar construction was first suggested by E. Specker in conversations with the second author in 2000, cf. [40, Section 7].

<sup>4</sup> This observation was suggested by T. Kotek in conversations with the second author in summer 2014.



- (ii) A nice logic  $\mathcal{L}$  has finite S-rank (P-rank, C-rank) iff all its definable properties have finite S-rank (P-rank, C-rank).

### Examples 7.

- (i) ([20]): FOL and CFOL have finite S-rank, C-rank and P-rank.  
(ii) ([20]): MSOL and CMSOL have finite S-rank and C-rank.  
(iii) The examples  $C_A, P_A$  above do not have finite S-rank.

### 3.3 Proof of Theorem 5

*Proof.* (i) The disjoint union of two graphs is never connected. Therefore all the entries of  $H(C_A \sqcup)$  are zero, unless we consider the empty graph to be structure. In this case we have exactly one row and one column representing  $C_A$ . In any case, the rank is  $\leq 2$ .

(ii) Consider the submatrix of  $H(C_A, \sqcup)$  consisting of rows and columns labeled with the edgeless graphs  $E_n$  and use Lemma 4.

(iii) We first observe that

- (\*) for any sum-like operation  $\square$  on  $\tau_{graphs}$ -structures (i.e., graphs),  $G$  and  $H$ , if  $G \square H = P_n$  for  $n \geq 3$ , either  $G$  or  $H$  must be the empty graph.

This is due to the fact that  $\tau_{graphs}$  has no constant symbols. Therefore, a row or column containing non-zero entries must be labeled by the empty graph.

(iv) Here we consider  $(\sigma, \tau)$ -translation schemes for sum-like operations, with  $\sigma = \tau_{graphs} \cup \{a\}$ . Hence (\*) from the proof of (iii) is not true anymore because now  $P_{m+n+1}$  can be obtained from  $P_n$  and  $P_m$  with the  $a$  being an end vertex, using  $\sqcup^1$ . So we apply Lemma 4.

(v) Connection operations of two large enough cliques still produce connected graphs, but never form a clique. Q.E.D.

### 3.4 Properties of Finite S-rank and Finite P-rank

Let  $\mathcal{S}(\tau)$  and  $\mathcal{P}(\tau)$  denote the collection of all  $\tau$ -properties of finite S-rank and finite P-rank respectively, and let  $\mathcal{S} = \bigcup_{\tau} \mathcal{S}(\tau)$  and  $\mathcal{P} = \bigcup_{\tau} \mathcal{P}(\tau)$ .

**Theorem 8.**  $\mathcal{S}$  and  $\mathcal{P}$  are abstract Lindström logics which have finite S-rank and finite P-rank, respectively.

*Sketch of proof:* One first gives  $\mathcal{S}$  and  $\mathcal{P}$  a canonical syntax as described in [31],[35]. The proof then is a tedious induction which will be published elsewhere. Q.E.D.

It is unclear whether the abstract Lindström logic  $\mathcal{S}$  goes beyond CMSOL. As of now, we were unable to find a  $\tau$ -property which has finite S-rank, but is not definable in CMSOL.

**Problem 2.**

- (i) *Is every  $\tau$ -property with finite S-rank definable in  $\text{CMSOL}(\tau)$ ?*
- (ii) *Is every  $\tau$ -property with finite P-rank definable in  $\text{CFOL}(\tau)$ ?*

It seems to us that the same can be shown for connection-like operations, but we have not yet checked the details.

## 4 Hankel matrices and the Feferman-Vaught theorem

### 4.1 The FV-property

In this section we look at nice Lindström logics with a fixed quantifier rank. We use it to derive from the classical Feferman-Vaught theorem an abstract version involving the quantifier rank. This differs from the treatment in [1, Chapter xviii]. Our purpose is to investigate the connection between Hankel matrices of finite rank and the Feferman-Vaught Theorem on finite structures in an abstract setting.

**Definition 9.** *Let  $\mathcal{L}$  be a nice logic with quantifier rank  $\rho$ .*

- (i) *We denote by  $\mathcal{L}(\tau)^q$  the set of  $\mathcal{L}(\tau)$ -sentences  $\phi$  (without free variables) with  $\rho(\phi) = q$ .*
- (ii) *Two  $\tau$ -structures  $\mathfrak{A}, \mathfrak{B}$  are  $\mathcal{L}^q$  equivalent,  $\mathfrak{A} \sim_{\mathcal{L}}^q \mathfrak{B}$ , if for every  $\phi \in \mathcal{L}(\tau)^q$  we have  $\mathfrak{A} \models \phi$  iff  $\mathfrak{B} \models \phi$ .*
- (iii)  *$\mathcal{L}$  has the FV-property for  $\square$  with respect to  $\rho$  if for every  $\phi \in \mathcal{L}(\tau)^q$  there are  $k = k(\phi) \in \mathbb{N}$ ,  $\psi_1, \dots, \psi_k \in \mathcal{L}(\tau)^q$  and  $B_\phi \in 2^{2^k}$  such that for all  $\tau$ -structures  $\mathfrak{A}, \mathfrak{B}$  we have that*

$$\mathfrak{A} \square \mathfrak{B} \models \phi$$

*iff*

$$B_\phi(\psi_1^A, \dots, \psi_k^A, \psi_1^B, \dots, \psi_k^B) = 1$$

*where for  $1 \leq j \leq k$  we have  $\psi_j^A = 1$  iff  $\mathfrak{A} \models \psi_j$  and  $\psi_j^B = 1$  iff  $\mathfrak{B} \models \psi_j$ .*

- (iv)  *$\square$  is  $\mathcal{L}$ -smooth with respect to  $\rho$  if for every two pairs of  $\tau$ -structures  $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$  with  $\mathfrak{A}_1 \sim_{\mathcal{L}}^q \mathfrak{A}_2$  and  $\mathfrak{B}_1 \sim_{\mathcal{L}}^q \mathfrak{B}_2$  we also have  $\mathfrak{A}_1 \square \mathfrak{B}_1 \sim_{\mathcal{L}}^q \mathfrak{A}_2 \square \mathfrak{B}_2$ .*

*If  $\rho$  is clear from the context we omit it.*

A close inspection of the classical proofs shows that the requirements concerning the quantifier rank are satisfied in the following cases.

### Examples 10.

- ([16]): FOL has the FV-property for all product-like and connection-like operations  $\square$ .
- ([25]): CFOL with quantifier rank  $\rho_2$  has the FV-property for all product-like and connection-like operations  $\square$ .
- ([22],[29],[45]): MSOL has the FV-property for all sum-like and connection-like operations  $\square$ .
- ([9]): CMSOL with quantifier rank  $\rho_2$  has the FV-property for all sum-like and connection-like operations  $\square$ .

## 4.2 The FV-property and Finite Rank

**Definition 11.** Let  $\mathcal{L}$  be a nice logic.

- (i) Let  $\square$  be a binary operation on  $\tau$ -structures.  $\mathcal{L}$  is  $\square$ -closed if all the equivalence classes of  $\equiv_{\Phi, \square}$  are definable in  $\mathcal{L}(\tau)$ .
- (ii)  $\mathcal{L}$  is  $S$ -closed ( $P$ -closed,  $C$ -closed) if for every sum-like (product-like, connection-like) binary operation  $\square$  the logic  $\mathcal{L}$  is  $\square$ -closed.

**Proposition 12.** Let  $\mathcal{L}$  have the FV-property for  $\square$ .

- (i)  $\square$  is  $\mathcal{L}$ -smooth.
- (ii) Let  $\Phi$  be a  $\tau$ -property definable by a formula  $\phi \in \mathcal{L}(\tau)^q$ . Then each equivalence class  $[\mathfrak{A}]_{\Phi, \square}$  of  $\equiv_{\Phi, \square}$  is definable by a formula  $\psi(\mathfrak{A}) \in \mathcal{L}(\tau)^q$ .
- (iii) If  $\mathcal{L}$  has the FV-property for all sum-like (product-like) operations then  $\mathcal{L}$  is  $S$ -closed ( $P$ -closed).

*Sketch of proof.* (i) Follows because for  $i = 1, 2$ , the truth value of  $\mathfrak{A}_i \square \mathfrak{B}_i \models \phi \in \mathcal{L}(\tau)^q$  depends only on  $B_\phi$ , the Boolean function associated with the FV-property.

(ii) Fix a  $\tau$ -structure  $\mathfrak{A}$ . We want to show that  $[\mathfrak{A}]_{\Phi, \square}$  is definable by some formula  $\psi(\mathfrak{A}) \in \mathcal{L}(\tau)^q$ .

$\mathfrak{B} \in [\mathfrak{A}]_{\square, \Phi}$  iff for all  $\mathfrak{C}$ ,  $\mathfrak{A} \square \mathfrak{C} \in \Phi$  iff  $\mathfrak{B} \square \mathfrak{C} \in \Phi$ .

We have, using  $B_\phi$ , that

$$\mathfrak{A} \equiv_{\Phi, \square} \mathfrak{B}$$

iff for all  $\mathfrak{C}$ ,

$$B_\phi(\psi_1^A, \dots, \psi_k^A, \psi_1^C, \dots, \psi_k^C) = B_\phi(\psi_1^B, \dots, \psi_k^B, \psi_1^C, \dots, \psi_k^C) \quad (3)$$

iff  $\forall X_1, \dots, X_k \in \{0, 1\}$ ,

$$B_\phi(\psi_1^A, \dots, \psi_k^A, X_1, \dots, X_k) = B_\phi(\psi_1^B, \dots, \psi_k^B, X_1, \dots, X_k) \quad (4)$$

where  $\psi_i^A$ ,  $\psi_i^B$  and  $\psi_i^C$  are as in Definition 9(iii). Equation (4) can be expressed by a formula  $\psi(\mathfrak{A}) \in \mathcal{L}(\tau)^q$ .

(iii) Follows from (ii).

Q.E.D.

By analyzing the proof in [20], one can prove:

**Theorem 13.** *Let  $\mathcal{L}$  be a nice Lindström logic with quantifier rank  $\rho$  and  $\square$  be a binary operation on  $\tau$ -structures. If  $\square$  is  $\mathcal{L}$ -smooth with respect to  $\rho$ , then every  $\mathcal{L}$ -definable  $\tau$ -property  $\Phi$  has finite  $\square$ -rank.*

*Sketch of proof.* Let  $\Phi$  be definable by  $\phi$  with quantifier rank  $\rho(\phi) = q$ . Now let  $\phi_i : i \leq \alpha \in \mathbb{N}$  be an enumeration of maximally consistent  $\mathcal{L}(\tau)^q$ -sentences (aka Hintikka sentences). By our assumption  $\rho$  is nice, so this is a finite set. Furthermore  $\phi$  is logically equivalent to a disjunction  $\bigvee_{i \in I} \phi_i$  with  $I \subseteq [\alpha]$ , any every  $\tau$ -structure satisfies exactly one  $\phi_i$ .

Now we use the smoothness of  $\square$ . If  $\mathfrak{A}, \mathfrak{B}$  are two  $\tau$ -structures satisfying the same  $\phi_i$ , then their rows (columns) in  $H(\Phi, \square)$  are identical. Hence the rank of  $H(\Phi, \square)$  is at most  $\alpha$ , or  $\alpha + 1$  when empty  $\tau$ -structures are allowed. Q.E.D.

Combining Theorem 13 with Proposition 12(i) we get:

**Corollary 14.** *Let  $\mathcal{L}$  be a nice Lindström logic which has the FV-property for the binary operation  $\square$ , and let  $\Phi$  be definable in  $\mathcal{L}$ . Then  $r(\Phi, \square)$  is finite.*

**Proposition 15.** *Let  $\mathcal{L}$  be a nice logic with quantifier rank  $\rho$  and  $\square$  be a fixed operation on  $\tau$ -structure, which is associative. Assume further that for every  $\phi \in \mathcal{L}(\tau)$ ,*

- (i) *the rank of  $H(\phi, \square)$  is finite, and*
- (ii) *all equivalence classes of  $\equiv_{\phi, \square}$  are definable with formulas of  $\mathcal{L}$  with quantifier rank  $\leq qr(\phi)$ .*

*Then  $\mathcal{L}$  has the FV-property for  $\square$ .*

We have now shown that  $\mathcal{L}$  having the FV-property for  $\square$  implies that  $\square$  is  $\mathcal{L}$ -smooth, and that smoothness implies finite rank, or equivalently, finite index.

In fact we have:

**Theorem 16.** *Let  $\mathcal{L}$  be a nice  $S$ -closed logic and let  $\square_1$  be a sum-like operation. Then the following are equivalent:*

- (i)  *$\mathcal{L}$  has the FV-property for every sum-like operation  $\square$ .*
- (ii)  *$\square_1$  is  $\mathcal{L}$ -smooth.*
- (iii) *For all  $\phi \in \mathcal{L}(\tau)$  and every sum-like  $\square$ , the  $\square$ -rank of  $\phi$  is finite.*
- (iv) *For all  $\phi \in \mathcal{L}(\tau)$  and every sum-like  $\square$ , the index of  $\equiv_{\phi, \square}$  is finite.*

*The same holds if we replace  $S$ -closed and sum-like by  $P$ -closed and product-like.*

*Proof.* (i) implies (ii) is Proposition 12.

(ii) implies (iii) is Theorem 13.

(iii) is equivalent to (iv) by Proposition 3.

Finally, (iii) implies (i) is Proposition 15.

Q.E.D.

## 5 The S-closure of a nice logic

Let  $\mathcal{L}$  be a nice logic of finite S-rank with quantifier rank  $\rho$ . We define  $Cl_S(\mathcal{L})$  to be the smallest Lindström logic such that for all sum-like

$$\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$$

and all  $Cl_S(\mathcal{L})$ -definable  $\tau$ -properties  $\Phi$ , all the equivalence classes of  $\equiv_{\Phi, \square}$  are also definable in  $Cl_S(\mathcal{L})$ . This gives us a Lindström logic which is S-closed. However, in order to be a nice logic, we have to extend  $\rho$  to  $\rho'$  in such a way that ensures it is still nice.

We proceed inductively. Recall that there are only finitely many sum-like operations  $\square$  for fixed  $\sigma$  and  $\tau$ . Let  $\ell(\sigma) = \sum_{R \in \sigma} r(R) + 1$  where  $R$  is a relation symbol of arity  $r(R)$  or a constant symbol of arity 0. Two vocabularies are *similar* if they have the same number of symbols of the same arity. The effect of a sum-like operation only depends on the similarity type of  $\sigma$  and  $\tau$ . Hence for fixed  $\ell(\sigma)$  and  $\ell(\tau)$ , there are only finitely many sum-like operations.

A typical step in the induction is as follows.

Given  $\mathcal{L}$  and  $\phi \in \mathcal{L}(\tau)^{\rho(\phi)}$  and a sum-like  $\square : \text{Str}(\sigma) \times \text{Str}(\sigma) \rightarrow \text{Str}(\tau)$ , there are only finitely many equivalence classes of  $\equiv_{\phi, \square}$ . Let  $E_i = E(\phi, \square)_i$  with  $i \leq \alpha = \alpha(\phi, \square)$  be a list of these equivalence classes.

We form  $\mathcal{L}'$  with quantifier rank  $\rho'$  as follows: If  $E_i$  is not definable in  $\mathcal{L}(\sigma)$  then we add it to  $\mathcal{L}$  using a Lindström quantifier with quantifier rank  $\rho'(E_i) = \rho(\phi) + \ell(\sigma) + \ell(\tau)$ .

$\mathcal{L}'$  is a Lindström logic. We have to show that  $\rho'$  is nice, i.e., for fixed  $q$  and fixed number of free variables,  $\mathcal{L}'(\tau)^q$  is finite up to logical equivalence. This follows from the fact that we only added finitely many Lindström quantifiers and that for all  $\phi \in \mathcal{L}$  we have that  $\rho'(\phi) = \rho(\phi)$ .

For our induction we start with  $\mathcal{L}_0 = \mathcal{L}$ .  $\mathcal{L}_1$  is obtained by doing the typical step for each  $\phi \in \mathcal{L}_0$  and each sum-like  $\square$ .  $\rho_1$  is the union of all quantifier rank functions of the previous steps. We still have to iterate this process by defining  $\mathcal{L}_j$  and  $\rho_j$  and take the limit.

We finally get:

**Theorem 17.** *Let  $\mathcal{L}$  be nice with quantifier rank  $\rho$  and of finite S-rank. Then  $Cl_S(\mathcal{L})$  with quantifier rank  $\rho'$  is nice and has the FV-property for all sum-like operations.*

The details will be published in [27].

## 6 Conclusions and open problems

At the beginning of this paper we asked whether one can characterize logics over finite structures which satisfy the Feferman-Vaught Theorem for the disjoint union, or more generally, for sum-like and product-like operations on  $\tau$ -structures. The purpose of this paper was to investigate new directions to attack

this problem, specifically by relating the Feferman-Vaught Theorem to Hankel matrices of finite rank. [Theorem 16](#) describes their exact relationship.

We also investigated under which conditions one can construct logics satisfying the Feferman-Vaught Theorem. [Theorem 5](#) shows that there are uncountably many  $\tau$ -properties which have finite rank Hankel matrices for specific sum-like operations. [Theorem 8](#) shows the existence of maximal Lindström logics  $\mathcal{S}$  and  $\mathcal{P}$  where all their definable  $\tau$ -properties have finite rank for all sum-like, respectively product-like, operations. However, we have no explicit description of these maximal logics.

**Problem 3.**

- (i) *Is every  $\tau$ -property with finite  $P$ -rank (or both finite  $P$ -rank and finite  $C$ -rank) definable in CFOL?*
- (ii) *Is every  $\tau$ -property with finite  $S$ -rank (finite  $C$ -rank) definable in CMSOL?*

In case the answers to the above are negative, we can ask:

**Problem 4.**

- (i) *How many  $\tau$ -properties are there with finite  $S$ -rank ( $P$ -rank,  $C$ -rank)?*
- (ii) *Is there a nice Gurevich logic where all the  $\tau$ -properties in  $\mathcal{S}$  are definable?*

In [40, Section 7, Conjecture 2] it is conjectured that there are continuum many nice Gurevich logics with the FV-property for the disjoint union. Adding  $C_A$  or  $P_A$  from [Theorem 5](#) for fixed  $A \subseteq \mathbb{N}$  as Lindström quantifiers to FOL together with all the equivalence classes of  $\equiv_{C_A, \sqcup}$  or  $\equiv_{P_A, \sqcup}$  gives us a nice Lindström logic. However, the definable  $\tau_{\text{graph}}$ -property that the complement of a graph  $G$  is in  $C_A$  has infinite  $\sqcup$ -rank, see [Theorem 5\(ii\)](#).

**Problem 5.** *How many different nice Gurevich logics with the FV-property for the disjoint union are there?*

A similar analysis for connection-like operations will be developed in [27].

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